

Hawking radiation from laser filaments?

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Belgiorno *et al* have reported on experiments aiming at the detection of (the analogue of) Hawking radiation using laser filaments [F. Belgiorno *et al*, Phys. Rev. Lett. **105**, 203901 (2010)]. They sent intense focused Bessel pulses into a non-linear dielectric medium in order to change its refractive index via the Kerr effect. We study this scenario with the parameters of their experiment and find that the effective metric does not quite correspond to black hole evaporation but obeys more similarities to cosmological particle creation – even though this effect can probably not quite explain the observations. Modelling Hawking radiation would require much stronger pulses.

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I. INTRODUCTION

The idea of finding analogies to gravity in laboratory systems is quite old. For example, already in 1923, W. Gordon used an effective metric in order to describe light propagation in media [1]. Later it has been realized [2] that these analogies could be employed to model quantum particle creation phenomena such as Hawking radiation [3, 4]. The Hawking effect is the thermal emission of radiation from horizons – either of black holes in general relativity, or in analogue systems, where such a horizon is that of the effective space-time metric defined by the wave equation for light (or other waves) propagating in the medium. An analogue of a black hole horizon can occur where the velocity of the medium exceeds the speed of light (or sound) within the medium, thereby dragging along the photons (or phonons).

By now there are many proposals for such analogue systems based on various laboratory systems, for instance sound propagation in flowing fluids, see, e.g., [5–8]. A black hole analogue for photons has been proposed in [9] employing the phenomenon of slow light – for a discussion of the prospects and difficulties of slow light, see, e.g., [10–13]. Partly motivated by these problems, black hole analogues based on the propagation of light in ordinary dielectrics were discussed in [14, 15]. An important idea was the realization that the motion of the medium can be replaced by the motion of a pulse through the medium which changes the local light speed, cf. [16].

Recently, Belgiorno *et al* [17] have argued that they have seen Hawking radiation from such an analog system, see also [18, 19]. Let us review their claim. They direct an intense pulse of radiation into a piece of silica glass, such that in a very thin region of finite size the Kerr effect increases the refractive index of the glass $n = n_0 + \delta n$. At a frequency ω separated from the fre-

quency of that incoming radiation, the phase velocity $v_{\text{ph}}(\omega)$ of light in the medium is such that outside the pulse $v_{\text{ph}}(\omega, n_0)$ is (slightly) greater than the speed of the pulse – while within that region, the phase velocity $v_{\text{ph}}(\omega, n_0 + \delta n)$ is slightly less than the pulse speed (due to the increased refractive index $n_0 + \delta n$). The point were the phase velocity (at that particular frequency) equals the velocity of the pulse is claimed to be a “phase horizon”. Looking at a direction perpendicular to the pulse, they see a slight excess of photons (about one photon every ten pulses) at this frequency where their phase horizon occurs. Because the observed radiation is thereby associated with this phase horizon, they claim it must be the Hawking effect.

In the following, we shall examine this claim in more detail. More precisely, the question we are going to examine is whether or not the existence of such phase horizons is enough for concluding the occurrence of Hawking radiation, and whether the existence of such an horizon can be responsible for the radiation they see. Note that the system studied in [17] is very different from most of the previously studied black hole analogues, since there is no low-frequency regime in which the phase and group velocities are equal and where the black hole horizon is unambiguously defined. A crucial feature of their system is that the dispersion relation of the electromagnetic field in the frequency regime of interest is far from trivial, and the group and phase velocities are different. Furthermore, there is no group velocity horizon (in the frequency range of interest), i.e., the group velocity of the photons is always smaller than the pulse speed.

Since it is hard to analyze a model in which the true dispersion relation of the silica glass [20] is taken into account [21], we shall assume that what is crucial in the argumentation in [17] is the existence of a phase – but not group – velocity horizon, and look at a model which shares this feature of their system although both, the cause of the dispersion relation, and the exact analytic form will be different. Because their claim (that they have seen Hawking radiation) rests crucially only on the existence of a phase horizon, the fact that this phase

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horizon is due to a different mechanism in our model from theirs should not make any difference if their argument was correct.

We shall use two models which behave very similarly to their model in the frequency/wave-number regime of interest. These models will have phase horizons in exactly their sense, with the same characteristic parameters – but no group velocity horizons in that regime, just as theirs does not. The models clearly have an altered effective space-time geometry, and one can ask what the quantum radiation is from that space-time and whether it could be classed as “Hawking radiation”.

II. EFFECTIVE METRIC

Let us start with the macroscopic Maxwell equations in a medium whose index of refraction n is purely determined by the relative dielectric permittivity $\varepsilon = n^2$, i.e., the relative magnetic permeability is unity $\mu = 1$. In natural units $\varepsilon_0 = \mu_0 = c_0 = \hbar = 1$, they read

$$\vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot (n^2 \vec{E}) = 0, \quad (1)$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad (2)$$

$$\vec{\nabla} \times \vec{H} = \vec{\nabla} \times \vec{B} = \partial_t \vec{D} = \partial_t (n^2 \vec{E}), \quad (3)$$

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}. \quad (4)$$

Using the temporal gauge $A_0 = 0$ (which is not equivalent to the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$ in this case), the two equations (2) and (4) can be satisfied automatically by introducing the usual vector potential via

$$\vec{E} = \partial_t \vec{A}, \quad (5)$$

$$\vec{B} = -\vec{\nabla} \times \vec{A}. \quad (6)$$

The remaining two Maxwell equations, i.e., Coulomb’s law (1) and Ampère’s law (3), then become

$$\vec{\nabla} \cdot (n^2 \vec{E}) = \vec{\nabla} \cdot (n^2 \partial_t \vec{A}) = 0, \quad (7)$$

$$\begin{aligned} \partial_t (n^2 \vec{E}) &= \partial_t (n^2 \partial_t \vec{A}) = \vec{\nabla} \times \vec{B} = -\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) \\ &= \nabla^2 \vec{A} - \vec{\nabla} (\vec{\nabla} \cdot \vec{A}). \end{aligned} \quad (8)$$

Let us now chose a specific polarization. For simplicity, we assume that n is a function of t and z only

$$n = n(t, z). \quad (9)$$

Thus the k_x and k_y , the eigenvalues of the operators $i\partial_x$ and $i\partial_y$, are constants – which allows us to use the separation ansatz $\vec{A}(t, \vec{r}) = \vec{A}(t, z) \exp\{ik_x x + ik_y y\}$. Given this ansatz, we can always rotate the system so that $k_y = 0$, and the solution $\vec{A}(t, \vec{r})$ is independent of y . We now choose the polarization with $A_x = A_z = 0$. Since $\vec{A}(t, \vec{r})$ does not depend on y , this automatically satisfies the Coulomb law (7) and Ampère’s law (8) becomes

$$\partial_t (n^2 \partial_t A) = \vec{\nabla}^2 A, \quad (10)$$

where $\vec{A} = A \vec{e}_y$ and $\vec{\nabla}$ has only x and z derivatives.

For the other polarization, it is more convenient to employ the dual potential $\vec{\Lambda}$ such that

$$\vec{D} = n^2 \vec{E} = \vec{\nabla} \times \vec{\Lambda}, \quad (11)$$

$$\vec{H} = \vec{B} = \partial_t \vec{\Lambda}. \quad (12)$$

This automatically obeys Coulomb’s law (1) and Ampère’s law (3). The remaining Maxwell equations read

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\partial_t \vec{\Lambda}) = 0, \quad (13)$$

$$\vec{\nabla} \times \vec{E} = \vec{\nabla} \times \left(\frac{1}{n^2} \vec{\nabla} \times \vec{\Lambda} \right) = -\partial_t \vec{B} = -\partial_t^2 \vec{\Lambda}. \quad (14)$$

We see that, for a purely dielectric medium without magnetic response, the description in terms of the dual potential $\vec{\Lambda}$ is actually simpler since it obeys the analogue of the Coulomb gauge $\vec{\nabla} \cdot \vec{\Lambda} = 0$ in view of Eq. (13). In complete analogy to the previous case, we select a solution $\vec{\Lambda}$ which is independent of y and, consistent with $\vec{\nabla} \cdot \vec{\Lambda} = 0$, we take $\Lambda_x = \Lambda_z = 0$. With these simplifications, Faraday’s law (14) can be cast into the form

$$\partial_t^2 \Lambda = \vec{\nabla} \cdot \left(\frac{1}{n^2} \vec{\nabla} \Lambda \right), \quad (15)$$

with $\vec{\Lambda}(t, \vec{r}) = \Lambda(t, x, z) \vec{e}_y$. We thus see that the two polarizations of the electromagnetic field do not obey the same wave equations (10) and (15) if n^2 actually depends on t or z (or both).

The wave equations (10) and (15) can be written as the equations of motion of a scalar field in an effective 2+1 dimensional (t, x, z) space-time. The effective metrics for the two polarizations are, cf. [22]

$$ds_A^2 = dt^2 - n^2(dx^2 + dz^2), \quad (16)$$

$$ds_\Lambda^2 = \frac{1}{n^4} ds_A^2, \quad (17)$$

where the first ds_A^2 corresponds to the equation (10) for A , and the second ds_Λ^2 to the equation (15) for Λ . The second effective metric is conformally related to the first so the light cones of both are the same. Unlike the four-dimensional equations for the electromagnetic field, however, these effective three-dimensional scalar equations are not conformally invariant and the two polarizations obey different equations of motion (for non-constant n).

III. DISPERSION

So far, we have described a dielectric medium without dispersion (where n depends on t and z , but not on ω , for example). In a real medium, however, the electric displacement $\vec{D}(t, \vec{r})$ is not just given by the electric field $\vec{E}(t, \vec{r})$ at that space-time point, but also depends on the electric field at earlier times. In a stationary medium,

this can be represented via $\vec{D}(\omega, \vec{r}) = \varepsilon(\omega, \vec{r})\vec{E}(\omega, \vec{r})$ after a temporal Fourier transformation. For a dynamical medium, however, this is no longer possible in general because of the time dependence of the dielectric constant.

In order to avoid these difficulties, we shall mimic dispersion by adding a mass term to the equations (10) and (15) of motion. As it is well known, a massive scalar field has a non-trivial dispersion relation, i.e., group and phase velocity depend on ω . This simple model has the advantage that the generalization to curved space-times is straight forward. Equation (10) is replaced by

$$\partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu A \right) + m^2 \sqrt{|g|} A = 0, \quad (18)$$

with the effective metric $g^{\mu\nu}$ given by (16) and analogously equation (15) for Λ is obtained with the effective line element (17).

Furthermore, as in the experiment [17], we shall assume that the small perturbation of the dispersion relation (created via the non-linear Kerr effect) moves through the medium with a constant velocity v (determined by the passage of an intense pulse of light through the medium). The two models we shall consider will differ in how that change is expressed. In the first model, the space-time dependence of the dispersion relation occurs through a variation of the refractive index

$$n = n(z - vt), \quad (19)$$

while the mass of the field is held constant. In the second model, the space-time dependence will come through a variation in the mass of the field

$$m = m(z - vt), \quad (20)$$

with the refractive index held constant. In the first model, the equation of motion (18) has the form

$$\left[\partial_t n^2 (z - vt) \partial_t - \partial_x^2 - \partial_z^2 + n^2 (z - vt) m_0^2 \right] A = 0. \quad (21)$$

In the second model, the refractive index is assumed to be constant $n = n_0$, but the mass m changes as a function of space-time

$$\left[\partial_t^2 - \frac{1}{n_0^2} (\partial_x^2 + \partial_z^2) + m^2 (x - vt) \right] A = 0. \quad (22)$$

In order to make this model correspond most closely to the experiment [17], we choose

$$\lim_{z^2 \rightarrow \infty} n = n_0 = 1.4595, \quad (23)$$

$$\lim_{z^2 \rightarrow \infty} m^2 = m_0^2 = \frac{0.208}{\mu\text{m}^2} = \left(\frac{2\pi}{13.46\mu\text{m}} \right)^2. \quad (24)$$

These factors were chosen so as to make the asymptotic effective refractive index $n_{\text{eff}}(\omega)$, given by the inverse of the phase velocity of the waves

$$n_{\text{eff}} = \frac{k}{\omega} = \frac{n_0}{\sqrt{1 + m_0^2 n_0^2 / k^2}} \approx n_0 \left[1 - \frac{m_0^2 n_0^2 \lambda^2}{8\pi^2} \right], \quad (25)$$

as close as possible to the actual refractive index measured in the experiment [17]. In particular the above values of n_0 and m_0 were chosen to fit the Silica glass refractive index at 700 and 1100 nm. Note that λ is the wavelength of the light in vacuum, not in the medium, i.e., it is really shorthand for $2\pi c_0/\omega$.

The wave equations for the other polarization read

$$\left[\partial_t^2 - \vec{\nabla} \cdot \frac{1}{n^2(z - vt)} \vec{\nabla} + \partial_x^2 + \frac{m_0^2}{n^4(z - vt)} \right] \Lambda = 0, \quad (26)$$

for the first model, and, for the second one,

$$\left[\partial_t^2 - \frac{1}{n_0^2} (\partial_x^2 + \partial_z^2) + \frac{m^2(z - vt)}{n_0^4} \right] \Lambda = 0. \quad (27)$$

Compared to the wave equations (21) and (22), the dispersion relation is the same with the mass term divided by n^4 . Thus, in order to fit the dispersion relation of the silica glass, we can use the same parameters for $n_0 = 1.459$ and take $m_A^2 = m_A^2 n_0^4 = 0.943/\mu\text{m}^2$.

Of course, Eq. (18) is simply the equation of motion of a massive scalar field in a background effective metric. As is well known, the phase velocity of a massive field is everywhere greater than the velocity of light (velocity of the characteristics of the metric), and the group velocity is everywhere lower, but the product is always equal to the velocity of light (in this case $c = 1/n_0$).

We are not saying that such an effective mass is the actual mechanism which determines the refractive index in the medium. It is clearly not. Rather it is the resonances of various electronic transitions which determine the refractive index of the system. However, our model mimics those features of the medium which the authors of [17] claim to be important to their explanation of the phenomenon they observe.

In particular, outside the band of wavelengths from 700 nm to 1100 nm this effective refractive index deviates significantly from that of Silica glass, and even within this band the second derivative of the refractive index with respect to the wavelength has the wrong sign, see Fig. 1. However, no-where in the analysis of [17] does the behavior of the refractive index outside this band, or the sign of the second derivative of the refractive index play a role. If they were important, that dependence would in itself case serious doubt on the attribution of the radiation measured in [17] to the Hawking effect.

IV. FIRST MODEL

In the experiment [17], the velocity of the pulse v is larger than the velocity $c = 1/n_0$ by a factor of $1.4590/1.4533 = 1.004$. Thus, we may apply a coordinate transformation (an effective Lorentz boost) after which the pulse is instantaneous. To this end, we define a new time coordinate (v is assumed to be constant)

$$\tau = t - \frac{z}{v}, \quad (28)$$

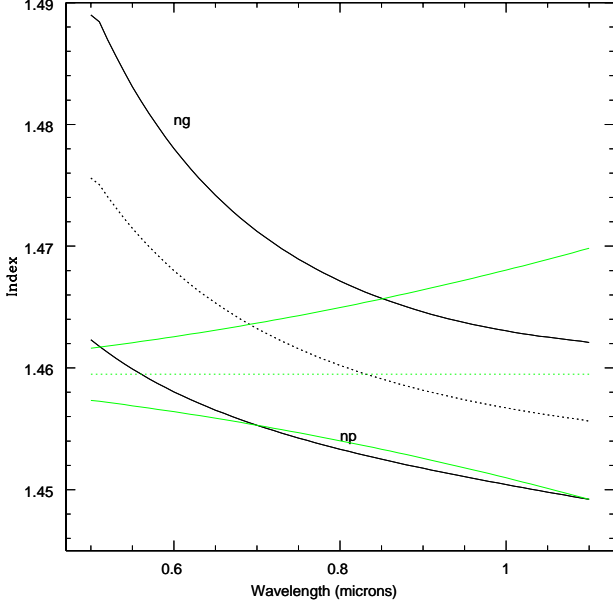


FIG. 1: Wavelength dependence of the effective refractive indices $n_{ph} = c/v_{ph}$, $n_{gr} = c/v_{gr}$, and $n_0 = c/\sqrt{v_{ph}v_{gr}}$ for silica glass (black) and for the massive approximation (green) which tries to fit the phase index of silica glass in the regime of interest by giving the field an effective mass. While this matches the phase index not too badly, it is worse for the group and geometric mean.

such that n depends on τ only $n = n(\tau)$. In order to obtain a diagonal metric, we also introduce a new spatial coordinate

$$\rho = z - \int d\tau \frac{v}{v^2 n^2(\tau) - 1}. \quad (29)$$

With the parameters from the experiment [17], the factor $v^2 n^2(\tau)$ is always greater than unity – even for the maximum value of δn : for $n = n_0 = 1.459$ and $v = c/1.4533$ as in [17], it is 1.008, whereas $\delta n \approx 0.001$.

Thus the above coordinate transformation is non-singular and results in a regular metric

$$ds_A^2 = \frac{n^2 v^2}{n^2 v^2 - 1} d\tau^2 - \frac{n^2 v^2 - 1}{v^2} d\rho^2 - n^2 dx^2, \quad (30)$$

as well as $ds_\Lambda^2 = ds_A^2/n^4$ for the other polarization. Remembering that n depends on τ only $n = n(\tau)$, we see that this metric is purely time-dependent and corresponds to an expanding or contracting Universe – without any black-hole horizon. Hence, at best one could argue that, if one saw particle creation in this system, one would have an analog to cosmological particle creation (see, e.g., [23, 24]), not black hole evaporation. There is

an effective anisotropic “Hubble parameter” given by

$$H_z(\tau) = \frac{1}{2} \frac{d \ln(g_{\rho\rho})}{d\tau} = \frac{n^2 v^2}{n^2 v^2 - 1} \frac{d \ln(n)}{d\tau}, \quad (31)$$

$$H_x(\tau) = \frac{d \ln(n)}{d\tau}. \quad (32)$$

Note that we are measuring the “Hubble parameter” using the time $d\tau$ which is related to the laboratory time dt , rather than the proper time ds of an observer at “rest” ($\rho = \text{const}$). Since $n^2 v^2$ is quite close to unity, the two “Hubble parameters” are very different $H_z \gg H_x$.

The equation of motion (for one polarization) reads

$$\left[\partial_\tau \frac{n^2 v^2 - 1}{v^2} \partial_\tau - \frac{n^2 v^2}{n^2 v^2 - 1} \partial_\rho^2 - \partial_x^2 + n^2 m^2 \right] A = 0 \quad (33)$$

Since ∂_ρ and ∂_x correspond to Killing vectors, i.e., symmetries of the metric, we may employ the separation ansatz $A(\tau, \rho, x) = A(\tau) \exp\{i\kappa\rho + ik_x x\}$ which gives

$$\left[\partial_\tau \frac{n^2 v^2 - 1}{v^2} \partial_\tau + \frac{n^2 v^2}{n^2 v^2 - 1} \kappa^2 + k_x^2 + n^2 m^2 \right] A = 0 \quad (34)$$

In order to study the solutions of this equation, it is useful to introduce yet another time coordinate via

$$T = \int d\tau \frac{v^2}{n^2 v^2 - 1}. \quad (35)$$

In terms of this time coordinate, each mode (κ, k_x) corresponds to a harmonic oscillator

$$\left(\frac{d^2}{dT^2} + \Omega^2(T) \right) A = 0, \quad (36)$$

with a time-dependent potential

$$\Omega^2 = n^2 \kappa^2 + (k_x^2 + n^2 m^2) \frac{n^2 v^2 - 1}{v^2}. \quad (37)$$

Again, since $n^2 v^2$ is quite close to unity, the first term $n^2 \kappa^2$ will dominate unless κ is very small. Before the pulse arrives, we have a constant potential

$$\Omega^2 = \Omega_0^2 = n_0^2 \kappa^2 + (k_x^2 + n_0^2 m^2) \frac{n_0^2 v^2 - 1}{v^2}, \quad (38)$$

and $\exp\{-i\Omega_0 T + i\kappa\rho + ik_x x\}$ is a solution. Translation to laboratory coordinates $\exp\{-i\omega t + ik_z z + ik_x x\}$ yields

$$\omega = \frac{v^2 \Omega_0 + v \kappa}{n_0^2 v^2 - 1}, \quad (39)$$

$$k_z = \frac{\Omega_0 + n_0^2 v^2 \kappa}{n_0^2 v^2 - 1}. \quad (40)$$

After the pulse, the potential is constant $\Omega^2 = \Omega_0^2$ again. However, a solution which initially behaves as $A = \exp\{-i\Omega_0 T + i\kappa\rho + ik_x x\}$ will after the pulse be a mixture of positive and negative frequencies in general

$$A = \alpha_{\kappa, k_x} e^{-i\Omega_0 T + i\kappa\rho + ik_x x} + \beta_{\kappa, k_x} e^{+i\Omega_0 T + i\kappa\rho + ik_x x}, \quad (41)$$

with the Bogoliubov coefficients α_{κ, k_x} and β_{κ, k_x} . The probability for particle creation (per mode κ, k_x) is given by $|\beta_{\kappa, k_x}^2|$. Since the variation $\delta\Omega$ of the potential is small, we may use perturbation theory to estimate the expected number of created photons per mode via [25, 26]

$$\beta_{\kappa, k_x} \approx \int dT \delta\Omega(T) \exp\{2i\Omega_0 T\}. \quad (42)$$

From Eq. (37), we obtain $\delta\Omega \sim \delta n$. Thus, in view of the smallness of $\delta n \approx 10^{-3}$, we can only hope for significant particle creation if the T -integration compensates this small number. If we assume the time-dependence of $\delta\Omega(T)$ to be pulse-like (e.g., Gaussian), this is only possible if Ω_0 is small enough, i.e., if $\delta\Omega$ constitutes a sizable fraction of Ω_0 . As discussed after Eq. (37), this requires a small κ . Thus, let us take $\kappa = 0$ in the following. Note that $\kappa = 0$ does not imply $k_z = 0$ according to Eq. (40). Setting $\kappa = 0$ and $k_x = 0$, we obtain the minimum value of Ω_0 to $n_0 m \sqrt{n_0^2 v^2 - 1}/v \approx n_0^2 m \sqrt{n_0^2 v^2 - 1}$ which yields

$$\Omega_0^{\min} \approx \frac{1}{11 \mu\text{m}} \approx \frac{2\pi}{70 \mu\text{m}} \rightarrow \mathcal{O}(10^{13} \text{Hz}). \quad (43)$$

In the experiment [17], $n(\tau)$ changes on time-scales $\Delta\tau$ of order $10\mu\text{m}$. However, after the transformation (35), we get a rescaled period ΔT of order $2500\mu\text{m}$. Consequently, since $\Omega_0^{\min} \Delta T \gg 1$ is very large, the particle creation amplitude (42) is exponentially suppressed by many orders of magnitude.

On the other hand, if $n(\tau)$ was to change on optical time-scales $\Delta\tau$ of order μm , i.e., femtosecond, we would get a rescaled period ΔT of a few hundred femtoseconds. Compared to the above value of Ω_0^{\min} , this would just be about the same order of magnitude – i.e., the particle creation probability would not be too strongly suppressed. Nevertheless, if we estimate the order of magnitude of

$$\delta\Omega \approx \frac{2\pi}{1300\mu\text{m}}, \quad (44)$$

we find that the probability $|\beta_{\kappa, k_x}^2|$ could be at the level of one percent at most. Still, one might hope to observe the analogue of cosmological particle creation with a set-up as in the experiment [17], provided that $n(\tau)$ changes fast enough. However, if we insert a transversal wavenumber in the optical regime, such as

$$k_x = \mathcal{O}\left(\frac{1}{n} \frac{2\pi}{\mu\text{m}}\right), \quad (45)$$

we increase Ω by a factor of around six and thus diminish the particle creation probability drastically (e.g., for a Gaussian pulse, several orders of magnitude). Consequently, the photons that are created would be emitted predominantly in forward direction and not to the side.

The analysis for the other polarization is very similar. Apart from the conformal factor $1/n^4$, the effective

metric ds_Λ^2 is the same as in Eq. (30). Thus the wave equation reads

$$\left[\partial_\tau \frac{n^2 v^2 - 1}{n^2 v^2} \partial_\tau - \frac{v^2}{n^2 v^2 - 1} \partial_\rho^2 - \frac{1}{n^2} \partial_x^2 + \frac{m^2}{n^4} \right] \Lambda = 0 \quad (46)$$

Again, we can introduce a new time coordinate

$$T = \int d\tau \frac{n^2 v^2}{n^2 v^2 - 1}. \quad (47)$$

and thereby map each mode to a harmonic oscillator with a time-dependent potential. Since the change in n is quite small $\delta n \approx 10^{-3}$ and mainly the relative change of $n^2 v^2 - 1$ matters, the additional factors of n^2 do not qualitatively affect our main conclusions.

V. SECOND MODEL

For the above model we have assumed that the mass of the field is constant in space and time, and that the change in the effective index of refraction $n(\omega) = |k|/\omega$ is caused by a change in the effective velocity of light $c = c_0/n$. We could also choose to have the change in the mass generate the change in the effective index of refraction. In this case that change would not be independent of λ but would scale as λ^2 . However we also know that the change in refractive index of silica glass due to the Kerr non-linearity will not be independent of λ , and nothing in the analysis of [17] was sensitive to any such dependence of the Kerr non-linearity on frequency. Thus our choosing such a variation should not take this model outside the range of conditions assumed in [17].

In the laboratory frame we have

$$n(\omega) = \frac{k}{\omega} = \frac{k}{\sqrt{k^2 + m^2}}, \quad (48)$$

so a change δm in the effective mass gives

$$\delta n = -\frac{km}{\sqrt{(k^2 + m^2)^3}} \delta m = -\frac{km}{\omega^3} \delta m. \quad (49)$$

Reproducing the variation $\delta n \approx 10^{-3}$ at optical frequencies $\omega = \mathcal{O}(2\pi/\mu\text{m})$ and wavenumbers k requires

$$\delta m \approx -\frac{2\pi}{100\mu\text{m}}, \quad (50)$$

for one polarization A , for the other Λ , it is about a factor of two larger. The remaining analysis is analogous to the previous section. Again, particle creation is most pronounced for $\kappa = k_x = 0$. The variation δm induces a small change of the effective potential

$$\delta\Omega \approx \frac{2\pi}{560\mu\text{m}}, \quad (51)$$

for one polarization A and $\delta\Omega \approx 2\pi/(280\mu\text{m})$ for the other Λ . Since the minimum value Ω_0^{\min} is basically the

same as in the previous section, we again find that the particle creation probability is negligible for the parameter $\Delta\tau = \mathcal{O}(10\mu\text{m})$ used in the experiment [17]. A measurable photon emission requires changing $m(\tau)$ on optical time-scales (or faster). In this case, the particle creation probability would be a bit higher than in the first model, but still not more than a few percent.

But clearly any particles created simply by a change δm in the effective mass of the propagating scalar particles is not the Hawking effect. This is related to our previous point, i.e., the absence of an effective black hole horizon. Instead, the photon creation mechanism is more similar to cosmological particle production. This effect can often be mapped to a time-dependent effective mass.

On the basis of the experiment [17], there is nothing to choose between either of the above models. But both of those models do not produce any particles due to an analogue of the Hawking effect. Furthermore, the number of particles created, and especially the number of particles created in a direction perpendicular (in the laboratory frame) to the propagation of the pulse (i.e., with $k_x \gg k_z$) are far too small to account for the observations in [17]. In addition, the created photons do not necessarily occur in the frequency band the authors of [17] observe.

VI. FILAMENTS

So far, we have assumed planar symmetry, i.e., n and m did only depend on t and z , but not on x or y . However, in the experiment [17], the travelling pulse is not a plane fronted wave, but is a thin filament. I.e., the pulse of changing n and m does not only depend on $t - z/v$ but also on x and y . In general the reduction of the Maxwell equations to an effective scalar field equation is much more difficult than in the planar case above. To simplify the analysis, we assume rotational symmetry which should be a reasonably good approximation. Thus, in cylindrical coordinates with $r = \sqrt{x^2 + y^2}$, we have

$$n = n(t, r, z) = n(t - z/v, r). \quad (52)$$

$$m = m(t, r, z) = m(t - z/v, r). \quad (53)$$

As the next step, we re-write Faraday's law (14) as

$$\partial_t^2 \vec{\Lambda} = -\vec{\nabla} \times \left(\frac{1}{n^2} \vec{\nabla} \times \vec{\Lambda} \right) = -\mathcal{D} \cdot \vec{\Lambda}, \quad (54)$$

where \mathcal{D} is a self-adjoint operator acting on the Hilbert space of all transversal $\vec{\nabla} \cdot \vec{\Lambda} = 0$ vector-valued functions $\vec{\Lambda}$. Since n does not depend on φ , this operator \mathcal{D} commutes with the generator $\hat{L}_z = i\partial_\varphi$ of rotations around the z -axis. Thus, we may classify the solutions of Eq. (54) in terms of $i\partial_\varphi$ eigenmodes.

In contrast to the planar case, where one could argue due to the rotational symmetry, that y -independence of

the modes could always be arranged (by suitable rotation), this is not true in this case. Any φ dependence cannot be eliminated by a coordinate transformation. However, the dominant photon creation would typically occur for the φ independent modes with only minor amounts from the modes with higher angular momentum. Thus, we focus on the φ independent and dominant solutions of Eq. (54) in the following.

We still have the freedom of selecting a polarization. If we choose $B_z = 0$ (analogous to a TM mode), we get $\Lambda_z = 0$ and from $\vec{\nabla} \cdot \vec{\Lambda} = 0$, we find $\Lambda_r = 0$. We are finally left with $\Lambda_\varphi(t, r, z)$ and after adding the effective mass term, Eq. (54) becomes

$$\left(\partial_t^2 - r\partial_r \frac{1}{n^2 r} \partial_r - \partial_z \frac{1}{n^2} \partial_z + \frac{m^2}{n^4} \right) \Lambda_\varphi = 0, \quad (55)$$

where we have added the effective mass term in order to model dispersion.

The other polarization with $E_z = 0$ (analogous to a TE mode) can be described in complete analogy in terms of the vector potential with $A_z = 0$. Again focussing on the dominant φ independent modes, $\vec{\nabla} \cdot (n^2 \partial_t \vec{A}) = 0$, i.e., Coulomb's law (7), implies $A_r = 0$. We are left with $A_\varphi(t, r, z)$ and Ampère's law (8) becomes

$$\left(\partial_t n^2 \partial_t - r\partial_r \frac{1}{r} \partial_r - \partial_z^2 + n^2 m^2 \right) A_\varphi = 0. \quad (56)$$

Note that we are using the co- and contra-variant notation $A^\varphi = A_\varphi/r^2$ and $\Lambda^\varphi = \Lambda_\varphi/r^2$ where Λ_φ and A_φ must go to zero as r^2 or faster in order that \vec{E} and \vec{B} be regular at $r = 0$.

The above equations (55) and (56) can again be interpreted as scalar wave equations in 3+1 dimensional curved space-times with the effective metrics

$$ds_A^2 = dt^2 - n^2 dr^2 - n^2 dz^2 - \frac{d\varphi^2}{r^2}, \quad (57)$$

$$ds_\Lambda^2 = \frac{dt^2}{n^4} - \frac{dr^2}{n^2} - \frac{dz^2}{n^2} - \frac{d\varphi^2}{r^2}, \quad (58)$$

provided that only φ independent solutions are considered. The weird $1/r^2$ dependence of the angular $d\varphi^2$ term stems from the re-interpretation of the double vector product as in Eq. (54) as a scalar Laplace operator. Again, making a suitable coordinate transformation

$$\tau = \frac{n_0 v}{\sqrt{n_0^2 v^2 - 1}} \left(t - \frac{z}{v} \right), \quad (59)$$

$$\rho = \frac{n_0^2 v z - t}{\sqrt{n_0^2 v^2 - 1}}, \quad (60)$$

we may diagonalize the unperturbed part of the metric. The resulting wave equation depends on the way how we model the change of the medium induced by the pulse – as a variation of the refractive index (first model) or via a change of the effective mass term (second model). Since the latter choice gave us a slightly larger probability for

particle creation in the previous Section and is technically simpler than the former, we use the second model in the following. From the transformed metric

$$ds_A^2 = d\tau^2 - d\rho^2 - n_0^2 dr^2 - \frac{d\varphi^2}{r^2}, \quad (61)$$

we obtain the wave equation

$$\left(\partial_\tau^2 - \partial_\rho^2 - \frac{r}{n_0^2} \partial_r \frac{1}{r} \partial_r + m^2(\tau, r) \right) A_\varphi = 0. \quad (62)$$

In analogy to the previous Sections, we use perturbation theory $m = m_0 + \delta m$ where the unperturbed modes can be obtained by the separation ansatz

$$A_{\Omega, \kappa, k_r}(\tau, \rho, r) = \mathcal{N}_{\Omega, \kappa, k_r} e^{-i\Omega\tau} e^{i\kappa\rho} r J_1(k_r r), \quad (63)$$

with the Bessel function J_1 . The Normalization factor $\mathcal{N}_{\Omega, \kappa, k_r}$ is chosen according to the pseudo norm

$$(A|A) = \frac{i}{2} \int d\rho dr \frac{2\pi}{r} (A^* \partial_\tau A - A \partial_\tau A^*), \quad (64)$$

discussed in the appendix, which gives

$$\mathcal{N}_{\Omega, \kappa, k_r} = \sqrt{\frac{k_r}{8\pi^3 \Omega}}. \quad (65)$$

The frequency Ω is given by

$$\Omega^2 = \frac{k_r^2}{n_0^2} + \kappa^2 + m_0^2. \quad (66)$$

Within perturbation theory (see appendix), the first-order amplitude $\mathcal{A}_{\Omega', \kappa', k'_r}^{\Omega, \kappa, k_r}$ for creating a photon with Ω, κ, k_r from an initial quantum vacuum fluctuation with Ω', κ', k'_r is given by the overlap integral

$$\mathcal{A}_{\Omega', \kappa', k'_r}^{\Omega, \kappa, k_r} \propto \int dr d\tau d\rho \frac{2m_0 \delta m}{r} A_{\Omega', \kappa', k'_r} A_{\Omega, \kappa, k_r}. \quad (67)$$

As the particle creation process is basically multi-mode squeezing, the same $\mathcal{A}_{\Omega', \kappa', k'_r}^{\Omega, \kappa, k_r}$ yields the amplitude for creating a pair of photons with the quantum numbers Ω, κ, k_r and Ω', κ', k'_r , respectively.

Since δm does not depend on ρ , the ρ integration yields $\delta(\kappa + \kappa')$. Thus, the particles are emitted in opposite ρ directions. Of course, this is strictly true only for an eternally propagating pulse $\delta m = \delta m(t - z/v, r)$, a finite life-time would also induce a weak dependence on ρ .

In the experiment [17], δm varies on time scales of the order of $10\mu\text{m}$. Again, the rate of change becomes even slower $\Delta\tau = \mathcal{O}(200\mu\text{m})$ after the transformation (59) and (60) to the new coordinates τ and ρ . In view of the value for $m_0 = 2\pi/(13.46\mu\text{m})$, the frequencies $\Omega + \Omega'$ of the photons to be emitted are far to large in comparison to $\Delta\tau$ and hence the integral over τ is exponentially suppressed by many orders of magnitude.

Similar to the planar case discussed before, we can only hope for a measurable particle creation probability if δm

varies on optical time scales (or faster). However, even in this case, significant particle creation is only possible for small enough κ and k_r . Consequently, the transversal wavenumber k_r should be far below the optical regime – i.e., optical photons are again predominantly emitted in forward direction. Since the transversal extent Δr of the pulse is a few microns at most – and thus much smaller than $2\pi/k_r$ – we may approximate the Bessel functions describing the radial dependence via $J_1(k_r r) \approx k_r r/2$. After that, the remaining power law dependence on k_r and k'_r can be pulled out of the integral and the total amplitude $\mathcal{A}_{\Omega', \kappa', k'_r}^{\Omega, \kappa, k_r}$ factorizes. The dependence on κ and κ' is given by the $\delta(\kappa + \kappa')$ term mentioned above. The remaining term depending on $\Omega + \Omega'$ reads

$$\mathcal{A}_{\Omega'}^\Omega \propto \int d\tau e^{-i(\Omega + \Omega')\tau} \int dr r^3 \delta m(\tau, r). \quad (68)$$

Thus, in comparison with the planar case discussed in the previous Sections, we find that the pair creation probability (per unit time) is additionally suppressed by a factor of $(\Delta r/\Delta\tau)^4$.

The other polarization (Λ) behaves in a very similar way – the equations are modified by factors of n_0 in various places. However, that does not change the order of magnitude estimated above.

VII. PULSE SUB-STRUCTURE

So far, we have considered a rigidly moving pulse, i.e., $n = n(t - z/v, r)$ and $m = m(t - z/v, r)$. However, the pulse in the refractive index created by the incoming $1.06\mu\text{m}$ radiation in [17] is not a simple (exponential) pulse, but contains sub-structure. The index of refraction change δn goes as the square of the incoming electric field strength $\delta n \propto E^2$, and since the incoming radiation is linearly, not circularly, polarized, the electric field E oscillates at the frequency of the incoming radiation, and the square E^2 then oscillates at twice that frequency. Taking this into account, a more realistic model of δn will then have the form (and similarly for δm)

$$\begin{aligned} \delta n(t, z, r) &= \delta n_0(r) \exp \left\{ -\frac{(t - z/v)^2}{2(\Delta t)^2} \right\} \times \\ &\times \cos^2 \left(\omega_{\text{in}} \left\{ t - \frac{z}{v_{\text{ph}}} \right\} \right). \end{aligned} \quad (69)$$

Here ω_{in} is the frequency of the incoming radiation (which generates the pulse) and v_{ph} its phase velocity – corrected by a factor of $1/\cos\Theta$, where $\Theta \approx 6.5^\circ$ is the cone angle of the Bessel pulse.

The oscillating pulse sub-structure given in the second line has two important consequences. First, in contrast to the pulse envelope, which is slowly varying on a time scale $\Delta t \geq \mathcal{O}(10\mu\text{m})$, the oscillating term is changing on optical time-scales (with $2\omega_{\text{in}}$), i.e., much faster. Second, since the pulse speed $v \approx c_0/1.453$ and the phase velocity

$v_{\text{ph}} \approx c_0/1.44$ do not coincide, the perturbation does not just depend on $\tau \propto (t - z/v)$, but also on ρ . Even though the difference between v and v_{ph} is only on the percent level, this deviation is enhanced by the coordinate transformation (59) and (60) to the τ, ρ coordinates

$$\delta n(\tau, \rho, r) = \delta n_0(r) \exp \left\{ -\frac{n_0^2 v^2 - 1}{2(n_0 v \Delta t)^2} \tau^2 \right\} \times \cos^2(\omega_\tau \tau + \omega_\rho \rho), \quad (70)$$

with the frequencies

$$\begin{aligned} \omega_\tau &= \omega_{\text{in}} \frac{\sqrt{n_0^2 v^2 - 1}}{n_0 v} \left(1 + \frac{1 - v/v_{\text{ph}}}{n_0^2 v^2 - 1} \right) \approx \frac{2\pi}{5\mu\text{m}}, \\ \omega_\rho &= \omega_{\text{in}} \frac{1 - v/v_{\text{ph}}}{\sqrt{n_0^2 v^2 - 1}} \approx \frac{2\pi}{10\mu\text{m}}. \end{aligned} \quad (71)$$

VIII. CONCLUSIONS

There are a number of conclusions one can draw from our analysis for the interpretation of the experiment [17].

First, whatever is causing the particle production due to the intense pulse, it is certainly not Hawking radiation caused by the “phase horizon”. The effective metric does not correspond to a black-hole horizon but has more similarities to a cosmological setting – similar to a purely super-luminal pulse, see also [27, 28]. Simulating a black-hole horizon would require pulses with other parameters, for example a stronger non-linearity δn , such that the effective metric in (30) is no longer regular.

Second, even the interpretation as cosmological particle creation yields negligible photon pair creation probabilities in the optical regime if we consider – as done in [17] – a rigidly moving pulse $\delta n(t - z/v, r)$ or $\delta n(t - z/v)$. After the effective Lorentz transformation, the rate of change of the pulse is just too slow. However, taking into account the oscillating sub-structure of the pulse (not just its envelope, as done in [17]), one could get a non-negligible probability for particle creation.

Third, in all of these scenarios, the creation of photons with transversal wave-numbers in the optical regime is strongly suppressed – almost all the photons are emitted roughly in longitudinal direction. Thus it is very hard to explain the photons emitted in perpendicular direction, as observed in [17]. If one were to include the effects of defects in the dielectric medium, this could change. On the one hand, they could effectively scatter the co-moving photons out of the co-moving direction. However, since it is hard to imagine more than a tiny fraction of the photons being so scattered, the photon creation rate by the pulse would have to be even larger to create enough photons. On the other hand, these defects could also induce short-scale deviations from a rigidly moving pulse $\delta n(t - z/v, r)$ with comparably high frequencies – similar to the pulse sub-structure discussed above – and thereby increase the particle creation rate.

Fourth, a key observation is that the particle production rate is not simply determined by the structure of the phase horizon. It depends in detail on how that phase horizon is created. As shown in the case of the “slab” geometry for the pulse, the two models for the variation of the material properties (first model versus second model) yield slightly different numbers of particles. This is another indication that this phase horizon is not an appropriate analog to the black hole horizons. In the latter case, the radiation (i.e., black hole evaporation) is solely a function of the structure of the horizon itself (i.e., the surface gravity). Here it depends in detail on how the horizon is modeled.

Fifth, it is also of importance that the equations of motion of the two polarizations (A_φ and Λ_φ) are not the same. While the effective metrics in the two cases are conformally related to each other, the wave equations for the polarizations are not conformally invariant. Were the effective metrics real 3+1 dimensional metrics, and the fields were the electromagnetic fields in 3+1 dimensions, the equations would be conformally invariant. However here the effective metrics are in reduced dimensions, and the equations for the potentials are effective scalar field equations. Both destroy the conformal invariance of the equations of motion. Thus, the propagation and particle production rates for the two polarizations will differ.

Finally, we find that more investigations are needed to determine the cause of the radiation observed in [17] and its potential relation to quantum radiation phenomena such as cosmological particle creation. We cannot claim to have explained their observations. This analog model clearly needs further work. Our modelling of the dispersion relation of light in such a dielectric medium is at best only a crude model of what takes place. Models which reproduce the dielectric behavior as encapsulated in the Sellmeier coefficients [20] are possible, and are being studied. However, we do not believe that they will alter our conclusions. Even were they to increase to the particle emission rate in a perpendicular direction, they would simply emphasize the dependence of the phenomenon on the details of the model, rather than relying solely on the structure of the horizon.

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Appendix

We wish to find the lowest order amplitude for the particle creation rate caused by a small fluctuation of the mass $\delta m(\tau, r)$ in the equation

$$\left(\partial_\tau^2 - \partial_\rho^2 - \frac{r}{n_0^2} \partial_r \frac{1}{r} \partial_r + m_0^2 + 2m_0 \delta m \right) \Psi = 0. \quad (72)$$

In terms of the momentum $\Pi = \partial_\tau \Psi$, the inner product

$$(\Psi_1 | \Psi_2) = \frac{i}{2} \int dr d\rho \frac{2\pi}{r} (\Psi_1^* \Pi_2 - \Pi_1^* \Psi_2), \quad (73)$$

is conserved $\partial_\tau (\Psi_1 | \Psi_2) = 0$ for any two solutions Ψ_1 and Ψ_2 . Introducing the matrices

$$\mathbf{M}_0 = \begin{bmatrix} 0 & 1 \\ \mathcal{D}^2 - m_0^2 & 0 \end{bmatrix}, \quad \delta \mathbf{M} = \begin{bmatrix} 0 & 0 \\ 2m_0 \delta m & 0 \end{bmatrix}, \quad (74)$$

where $\mathcal{D}^2 = \partial_\rho^2 + n_0^{-2} r \partial_r r^{-1} \partial_r$ is the spatial differential operator in Eq. (72), the equations of motion can be cast into the form

$$\partial_\tau \begin{bmatrix} \Psi \\ \Pi \end{bmatrix} = (\mathbf{M}_0 + \delta \mathbf{M}) \cdot \begin{bmatrix} \Psi \\ \Pi \end{bmatrix}. \quad (75)$$

Now, let (Ψ_I, Π_I) be a complete set of solutions of the unperturbed wave equation (72) with $\delta m = 0$ which have positive pseudo norm and are orthogonal $(\Psi_I | \Psi_J) = \delta_{IJ}$ with respect to the inner product (73), and such that $(\Psi_I | \Psi_J^*) = 0$. For simplicity, we choose the (Ψ_I, Π_I) as eigenvectors of the matrix operator \mathbf{M}_0 with eigenvalues $-i\Omega_I$. Since \mathbf{M}_0 is real, the complex conjugated solutions (Ψ_I^*, Π_I^*) then form a complete set of negative pseudo norm $(\Psi_I^* | \Psi_J^*) = -\delta_{IJ}$. Thus we can expand the solution (Ψ, Π) of the full wave equation (72) with $\delta m \neq 0$ into these sets

$$\begin{bmatrix} \Psi \\ \Pi \end{bmatrix} = \sum_I \left(\alpha_I(\tau) \begin{bmatrix} \Psi_I \\ \Pi_I \end{bmatrix} + \beta_I(\tau) \begin{bmatrix} \Psi_I^* \\ \Pi_I^* \end{bmatrix} \right). \quad (76)$$

Initially $\tau \rightarrow -\infty$, i.e., before the pulse arrives, the solution (Ψ, Π) is supposed to coincide with the unperturbed mode $I = 0$. Thus we have $\alpha_0(\tau \rightarrow -\infty) = 1$ while all other α_I as well as all β_I vanish for $\tau \rightarrow -\infty$. In the final regime $\tau \rightarrow \infty$, however, some of these modes will be excited $\beta_I = \mathcal{O}(\delta m)$ and $\alpha_{I \neq 0} = \mathcal{O}(\delta m)$ due to the perturbation $\delta \mathbf{M}$ induced by the pulse. Inserting the expansion (76) into the equations of motion (75) and neglecting terms of $\mathcal{O}([\delta m]^2)$, we find

$$\sum_I \left(\begin{bmatrix} \Psi_I \\ \Pi_I \end{bmatrix} \partial_\tau \alpha_I + \begin{bmatrix} \Psi_I^* \\ \Pi_I^* \end{bmatrix} \partial_\tau \beta_I \right) \approx 2m_0 \delta m \begin{bmatrix} 0 \\ \Psi_0 \end{bmatrix} \quad (77)$$

Projecting the the above equation onto the mode J via the inner product (73), we get

$$\partial_\tau \beta_J = -im_0 \int dr d\rho \frac{2\pi}{r} \delta m \Psi_J \Psi_0 + \mathcal{O}([\delta m]^2). \quad (78)$$

Integration over τ yields Eq. (67)

$$\beta_J(\tau \rightarrow \infty) \approx -im_0 \int d\tau dr d\rho \frac{2\pi}{r} \delta m \Psi_J \Psi_0. \quad (79)$$

This gives the amplitude for converting an initial quantum vacuum fluctuation in the mode Ψ_0 into a real particle in the final mode Ψ_J . The total probability for creating a particle in the final mode J is then given by the sum over all initial modes I

$$\mathcal{P}_J = \sum_I |\beta_{IJ}|^2 = m_0^2 \sum_I \left| \int d\tau dr d\rho \frac{2\pi}{r} \delta m \Psi_I \Psi_J \right|^2 \quad (80)$$

For our system, this becomes

$$\begin{aligned} \beta_{IJ} &= \mathcal{A}_{\Omega', \kappa', k_r'}^{\Omega, \kappa, k_r} = -\frac{im_0}{4\pi} \sqrt{\frac{k_r k_r'}{\Omega \Omega'}} \int d\tau dr d\rho r \delta m(\tau, r) e^{-i\Omega\tau} e^{i\kappa\rho} J_1(k_r r) e^{-i\Omega'\tau} e^{i\kappa'\rho} J_1(k_r' r) \\ &= -\frac{im_0}{2} \sqrt{\frac{k_r k_r'}{\Omega \Omega'}} \delta(\kappa + \kappa') \int d\tau e^{-i(\Omega + \Omega')\tau} \int dr r \delta m(\tau, r) J_1(k_r r) J_1(k_r' r). \end{aligned} \quad (81)$$

Calculating the total probability $\mathcal{P}_{\Omega, \kappa, k_r}$ then gives the usual $\delta(0)$ type singularity from the κ' integral over $[\delta(\kappa + \kappa')]^2$ because of the assumed unboundedness of the pulse in the ρ direction. Removing it gives the particle emission rate per unit length of the pulse in the ρ direction. Note

that Ω and Ω' are both positive, as we have chosen modes going as $e^{-i\Omega\tau}$ as our set of positive pseudo norm modes. The Bessel functions $J_1(k_r r)$ go as $k_r r/2$ when small, and become oscillatory for large $k_r r$.

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- | | |
|------|--------------|
| $B1$ | .473115591 |
| $B2$ | .631038719 |
| $B3$ | .906404498 |
| $C1$ | .0129957170 |
| $C2$ | .00412809220 |
| $C3$ | 98.7685322 |
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